Derivatives:

3.2 Differentiation Rules

The process of calculating a derivative is called differentiation. The goal of this section is to show how to differentiate functions rapidly—without having to apply the definition each time. It will then be an easy matter to calculate the velocities, accelerations, and other important rates of change we will encounter in Section 3.3.

Integer Powers, Multiples, Sums, and Differences

The first rule of differentiation is that the derivative of every constant function is zero. In short,

RULE 1

Derivative of a Constant

If c is a constant, then

$$\frac{d}{dx}(c) = 0.$$

EXAMPLE1: $\frac{d}{dx}(6) = 0$

Rule 2:

Power Rule for Positive Integer Powers of x

If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

To apply the power rule, we subtract 1 from the original exponent (n) and multiply the result by n.

EXAMPLE 2:

$$\frac{d}{dx}(x) = \frac{d}{dx}(x^1) = 1 \cdot x^0 = 1$$

$$\frac{d}{dx}(x^2) = 2x^1 = 2x$$

$$\frac{d}{dx}(x^3) = 3x^2$$

$$\frac{d}{dx}(x^4) = 4x^3$$

RULE 3 and 4:

The sum and difference rule :

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are differentiable functions of $\boldsymbol{x}$ then their sum and difference are differentiable at every point where $\boldsymbol{u}$ and $\boldsymbol{v}$ are both differentiable . At such point,

1. $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$
2. $\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$

Similar equations hold for more than two functions, as long as the number of functions involved is finite .

EXAMPLE 3:

1. $y = x^4 + 12x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x)$$

$$= 4x^3 + 12$$

EXAMPLE 4: $y = \frac{7x^2}{3} - 5 \quad \rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{7x^2}{3}\right) - \frac{d}{dx}(5) = \frac{7}{3} * 2x - 0 = \frac{14}{3}x$

Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is *not* the product of their derivatives. For instance,

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x, \qquad \text{while} \qquad \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1.$$

The derivative of a product of two functions is the sum of *two* products, as we now explain.

Rule 5 The Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

The derivative of the product uv is u times the derivative of v plus v times the derivative of u. In *prime notation*, $(uv)' = uv' + vu'$.

EXAMPLE 7 Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Solution From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\begin{aligned}\frac{d}{dx}[(x^2+1)(x^3+3)] &= (x^2+1)(3x^2) + (x^3+3)(2x)\\ &= 3x^4 + 3x^2 + 2x^4 + 6x\\ &= 5x^4 + 3x^2 + 6x.\end{aligned}$$

❑

There are times, however, when the Product Rule *must* be used. In the following example, we have only numerical values to work with.

EXAMPLE : Let $y = uv$ be the product of the functions u and v. Find $y'(2)$ if

$$u(2) = 3, \quad u'(2) = -4, \quad v(2) = 1, \quad \text{and} \quad v'(2) = 2.$$

Solution From the Product Rule, in the form

$$y' = (uv)' = uv' + vu',$$

we have

$$\begin{aligned} y'(2) &= u(2)v'(2) + v(2)u'(2) \\ &= (3)(2) + (1)(-4) = 6 - 4 = 2. \end{aligned}$$

Quotients

Just as the derivative of the product of two differentiable functions is not the product of their derivatives, the derivative of the quotient of two functions is not the quotient of their derivatives. What happens instead is this:

Rule 6 The Quotient Rule

If u and v are differentiable at x, and $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

EXAMPLE 6 Find the derivative of $y = \dfrac{t^2 - 1}{t^2 + 1}$.

Solution We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^2 + 1$:

$$\frac{dy}{dt} = \frac{(t^2+1) \cdot 2t - (t^2-1) \cdot 2t}{(t^2+1)^2} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}$$

$$= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2}$$

$$= \frac{4t}{(t^2+1)^2}.$$

❑

The Power Rule for Negative Integers

The Power Rule for negative integers is the same as the rule for positive integers.

> **Rule 7 Power Rule for Negative Integers**
>
> If n is a negative integer and $x \neq 0$, then
>
> $$\frac{d}{dx}(x^n) = nx^{n-1}.$$

EXAMPLE 7

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{4}{x^3}\right) = 4\frac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\frac{12}{x^4}$$

(3-4) Derivative of Trigonometric Function:

GENERALIZED DERIVATIVE FORMULAS

$\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$ (n an integer)	$\frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}}\frac{du}{dx}$
$\frac{d}{dx}[\sin u] = \cos u\frac{du}{dx}$	$\frac{d}{dx}[\cos u] = -\sin u\frac{du}{dx}$
$\frac{d}{dx}[\tan u] = \sec^2 u\frac{du}{dx}$	$\frac{d}{dx}[\cot u] = -\csc^2 u\frac{du}{dx}$
$\frac{d}{dx}[\sec u] = \sec u \tan u\frac{du}{dx}$	$\frac{d}{dx}[\csc u] = -\csc u \cot u\frac{du}{dx}$

Example 3 Find

(a) $\frac{d}{dx}[\sin(2x)]$ (b) $\frac{d}{dx}[\tan(x^2+1)]$ (c) $\frac{d}{dx}[\sqrt{x^3+\csc x}]$

(d) $\frac{d}{dx}[(1+x^5\cot x)^{-8}]$ (e) $\frac{d}{dx}\left[\frac{1}{x^3+2x-3}\right]$

Solution (a). Taking $u = 2x$ in the generalized derivative formula for $\sin u$ yields

$$\frac{d}{dx}[\sin(2x)] = \frac{d}{dx}[\sin u] = \cos u\frac{du}{dx} = \cos 2x \cdot \frac{d}{dx}[2x] = \cos 2x \cdot 2 = 2\cos 2x$$

Solution (b). Taking $u = x^2+1$ in the generalized derivative formula for $\tan u$ yields

$$\frac{d}{dx}[\tan(x^2+1)] = \frac{d}{dx}[\tan u] = \sec^2 u\frac{du}{dx}$$

$$= \sec^2(x^2+1)\cdot\frac{d}{dx}[x^2+1] = \sec^2(x^2+1)\cdot 2x$$

$$= 2x\sec^2(x^2+1)$$

Solution (c). Taking $u = x^3+\csc x$ in the generalized derivative formula for $\sqrt{u}$ yields

$$\frac{d}{dx}[\sqrt{x^3+\csc x}] = \frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}}\frac{du}{dx} = \frac{1}{2\sqrt{x^3+\csc x}}\cdot\frac{d}{dx}[x^3+\csc x]$$

$$= \frac{1}{2\sqrt{x^3+\csc x}}\cdot(3x^2-\csc x\cot x) = \frac{3x^2-\csc x\cot x}{2\sqrt{x^3+\csc x}}$$

Solution (d). Taking $u = 1+x^5\cot x$ in the generalized derivative formula for u^{-8} yields

$$\frac{d}{dx}[(1+x^5\cot x)^{-8}] = \frac{d}{dx}[u^{-8}] = -8u^{-9}\frac{du}{dx}$$

$$= -8\left(1+x^5\cot x\right)^{-9}\cdot\frac{d}{dx}[1+x^5\cot x]$$

$$= -8\left(1+x^5\cot x\right)^{-9}\cdot(x^5(-\csc^2 x)+5x^4\cot x)$$

$$= (8x^5\csc^2 x - 40x^4\cot x)\left(1+x^5\cot x\right)^{-9}$$

Solution (e). Taking $u = x^3 + 2x - 3$ in the generalized derivative formula for u^{-1} yields

$$\frac{d}{dx}\left[\frac{1}{x^3+2x-3}\right] = \frac{d}{dx}[(x^3+2x-3)^{-1}] = \frac{d}{dx}[u^{-1}]$$
$$= -u^{-2}\frac{du}{dx} = -(x^3+2x-3)^{-2}\frac{d}{dx}[x^3+2x-3]$$
$$= -(x^3+2x-3)^{-2}(3x^2+2) = -\frac{3x^2+2}{(x^3+2x-3)^2}$$

Example 5 From (15)

$$\frac{d}{dx}[x^{4/5}] = \frac{4}{5}x^{(4/5)-1} = \frac{4}{5}x^{-1/5}$$
$$\frac{d}{dx}[x^{-7/8}] = -\frac{7}{8}x^{(-7/8)-1} = -\frac{7}{8}x^{-15/8}$$
$$\frac{d}{dx}[\sqrt[3]{x}] = \frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$

(3-5) Higher Derivatives:

- If the derivative f' of a function f is itself differentiable, then the derivative of f' is denoted by f'' and is called the ***second derivative*** of f. As long as we have differentiability, we can continue the process of differentiating derivatives to obtain third, fourth, fifth, and even higher derivatives of f. The successive derivatives of f are denoted by

$$f', \quad f''=(f')', \quad f'''=(f'')', \quad f^{(4)}=(f''')', \quad f^{(5)}=(f^{(4)})', \ldots$$

These are called the first derivative, the second derivative, the third derivative, and so forth. Beyond the third derivative, it is too clumsy to continue using primes, so we switch from primes to integers in parentheses to denote the ***order*** of the derivative. In this notation it is easy to denote a derivative of arbitrary order by writing

$f^{(n)}$ The nth derivative of f

The significance of the derivatives of order 2 and higher will be discussed later.

Example 8 If $f(x)=3x^4-2x^3+x^2-4x+2$, then

$$\begin{aligned}
f'(x) &= 12x^3-6x^2+2x-4\\
f''(x) &= 36x^2-12x+2\\
f'''(x) &= 72x-12\\
f^{(4)}(x) &= 72\\
f^{(5)}(x) &= 0\\
&\vdots\\
f^{(n)}(x) &= 0 \quad (n \geq 5)
\end{aligned}$$

Example 3 Find $y''(\pi/4)$ if $y(x)=\sec x$.

Solution.

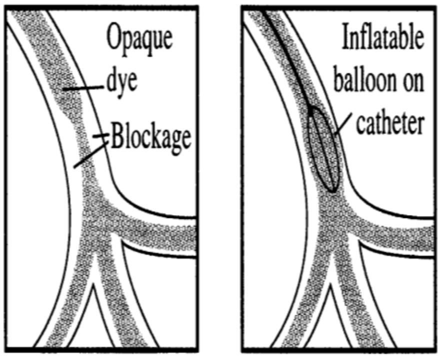
$$\begin{aligned}
y'(x) &= \sec x \tan x\\
y''(x) &= \sec x \cdot \frac{d}{dx}[\tan x] + \tan x \cdot \frac{d}{dx}[\sec x]\\
&= \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x\\
&= \sec^3 x + \sec x \tan^2 x
\end{aligned}$$

Thus,

$$\begin{aligned}
y''(\pi/4) &= \sec^3(\pi/4) + \sec(\pi/4)\tan^2(\pi/4)\\
&= (\sqrt{2})^3 + (\sqrt{2})(1)^2 = 3\sqrt{2}
\end{aligned}$$

EXAMPLE * *Relief from a heart attack.* A heart attack victim has been given a blood vessel dilator to lower the pressure against which the heart has to pump. For a short while after the drug is administered, the radii of the affected blood vessels will increase at about 1% per minute. According to Poiseuille's law, $V = kr^4$ (Section 3.7, Example 10), what percentage rate of increase can we expect in the blood flow over the next few minutes (all other things being equal)?

Angiography: An opaque dye is injected into a partially blocked artery to make the inside visible under x-rays. This reveals the location and severity of the blockage.



Angioplasty: A balloon-tipped catheter is inflated inside the artery to widen it at the blockage site.

EXAMPLE * *Unclogging arteries*

In the late 1830s, the French physiologist Jean Poiseuille ("pwa-*zoy*") discovered the formula we use today to predict how much the radius of a partially clogged artery has to be expanded to restore normal flow. His formula,

$$V = kr^4,$$

says that the volume V of fluid flowing through a small pipe or tube in a unit of time at a fixed pressure is a constant times the fourth power of the tube's radius r. How will a 10% increase in r affect V?

Solution The differentials of r and V are related by the equation

$$dV = \frac{dV}{dr}dr = 4kr^3 dr.$$

Hence,

$$\frac{dV}{V} = \frac{4kr^3 dr}{kr^4} = 4\frac{dr}{r}. \qquad \text{Dividing by } V = kr^4$$

The relative change in V is 4 times the relative change in r, so a 10% increase in r will produce a 40% increase in the flow. ❑